

On Conway and Kochen's “Thou shalt not clone one bit!”

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A physicists' explanation of why thou shalt not clone one bit gives a stronger version of Conway and Kochen's theorem.

John Conway and Simon Kochen prove in arXiv:0711.2310 [quant-ph] that it is impossible to have three spins-1, of which two are “twinned”, while the third necessarily gives the same answer as those two do to a “spin-zero measurement” along one given direction. A *spin-zero measurement* along \mathbf{n} is a measurement that asks whether or not the spin along the direction \mathbf{n} is zero. Two spins-1 are *twinned* if each gives the same answer (yes or no) to a spin-zero measurement along \mathbf{n} , regardless of the choice of \mathbf{n} .

For example, two spin-1 particles in the singlet (zero total angular momentum state)

$$|\Phi\rangle = (1/\sqrt{3})(|0\rangle|0\rangle - |1\rangle|-1\rangle - |-1\rangle|1\rangle) \quad (1)$$

are twinned in the Conway-Kochen sense, because $|\Phi\rangle$ has the form (1) regardless of the common direction \mathbf{n} along which the two spin components are specified. In this particular case a stronger form of the Conway-Kochen theorem follows immediately from the fact that a system in a *pure state* (in this case the two twinned spins) can have *no correlations whatever with any external system* (in this case the third spin). A proof of this elementary but fundamental property of pure states, a pillar of the “Ithaca Interpretation of Quantum Mechanics”, can be found in Appendix B of arXiv:quant-ph/9801057.

It is plausible to conjecture that requiring the twin condition to hold for *arbitrary* directions \mathbf{n} is so strong a constraint that it can happen *only* in the rotationally invariant singlet state. The stronger Conway-Kochen theorem would then follow immediately from the general fact that a system in a pure state can have no external correlations. In the remainder of this note I establish this conjecture with a simple but slightly unconventional application of elementary angular momentum technology.

Call the angular momentum vector operators (in units of \hbar) for the two spins-1 \mathbf{L} and \mathbf{S} . The twin condition is the requirement that the measured squared spin components of the two spins are both 0 or both 1, regardless of the common direction \mathbf{n} along which they are measured. The analytical expression of the twin condition is

$$0 = \text{Tr}(\rho[(\mathbf{n} \cdot \mathbf{L})^2 - (\mathbf{n} \cdot \mathbf{S})^2]^2) \quad (2)$$

where ρ is the density matrix for the two twinned spins. We must prove that (2) can hold for arbitrary directions \mathbf{n} only if ρ is the projection operator on the singlet state. It follows from (2) that any eigenstate $|\Psi\rangle$ of ρ associated with a non-zero (necessarily

positive) eigenvalue must satisfy

$$[(\mathbf{n} \cdot \mathbf{L})^2 - (\mathbf{n} \cdot \mathbf{S})^2]|\Psi\rangle = 0. \quad (3)$$

We now prove that the only $|\Psi\rangle$ satisfying (3) for all directions \mathbf{n} is the singlet state, and therefore ρ must indeed be the singlet-state projection operator.

Eq. (3) is equivalent to the requirement that

$$(S_i S_j + S_j S_i)|\Psi\rangle = (L_i L_j + L_j L_i)|\Psi\rangle \quad (4)$$

for all the components of \mathbf{L} and \mathbf{S} along any three orthogonal directions $i, j = x, y, z$. Act on both sides of (4) with $L_i L_j$, sum on i and j , use the fact that

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 = 2 = \mathbf{S}^2 = S_x^2 + S_y^2 + S_z^2 \quad (5)$$

and the standard angular-momentum commutation relations

$$L_x L_y - L_y L_x = i L_z, \text{ et cyc.},$$

$$S_x S_y - S_y S_x = i S_z, \text{ et cyc.},$$

$$L_i S_j - S_j L_i = 0, \quad (6)$$

to get from (4)

$$[2(\mathbf{L} \cdot \mathbf{S})^2 + (\mathbf{L} \cdot \mathbf{S})]|\Psi\rangle = [2(\mathbf{L}^2)^2 - (\mathbf{L}^2)]|\Psi\rangle = 6|\Psi\rangle. \quad (7)$$

Now note that for two spins-1 the 9-dimensional space of two-spin states $|\Psi\rangle$ is the direct sum of 5-dimensional ($j = 2$), 3 dimensional ($j = 1$), and 1-dimensional ($j = 0$, singlet) subspaces of total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Since in each such subspace

$$j(j+1) = \mathbf{J}^2 = \mathbf{L}^2 + 2\mathbf{L} \cdot \mathbf{S} + \mathbf{S}^2 = 4 + 2\mathbf{L} \cdot \mathbf{S}, \quad (8)$$

the states in each of the three subspaces are eigenstates of $\mathbf{L} \cdot \mathbf{S}$ with eigenvalues

$$1 \ (j = 2), \ -1 \ (j = 1), \ -2 \ (j = 0). \quad (9)$$

The projection operators on these subspaces commute with $\mathbf{L} \cdot \mathbf{S}$. Therefore if we act on (7) with projection operators on the $j = 2$, $j = 1$, and $j = 0$ subspaces then (9) requires the projection of $|\Psi\rangle$ on the $j = 2$ and $j = 1$ subspaces to vanish, but allows the projection of $|\Psi\rangle$ on the (one dimensional) $j = 0$ subspace not to vanish. So the two-spin density matrix ρ for twinned spins must indeed be a projection operator on the singlet state, and measurement outcomes on any third spin (or any other external system) must therefore be statistically independent of any measurement outcomes on the twins.